Adaptive widths of the approximation functional on the Sobolev space W_2^r with Gaussian measure*

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Abstract The tight orders for the Kolmogorov and adaptive (n, ε, δ)-widths of the Sobolev spaces W_2^r equipped with a Gaussian measure in the L_1-norm and L_∞-norm are determined by the method of discretization, which is based on reducing the calculation of the (n, ε, δ)-widths of the Sobolev space to the calculation of (n, ε, δ)-widths of finite-dimensional set equipped with the Gaussian measure.

Keywords: Kolmogorov (n, ε, δ)-widths, non-adaptive and adaptive, Gaussian measure.

In this paper, as the continuance of Maiorov's recent work[1], we investigate the distribution of values of the approximation functional and the Kolmogorov and adaptive probabilistic (n, ε, δ)-widths defined on Sobolev spaces equipped with a Gaussian measure. First we recall some definitions.

Let X be a separable Banach space and W some compact set in X. Let Y be a subspace of X and $Y^n = Y \times \cdots \times Y$ the product of n copies of Y. We denote the elements of Y^n by $y = (y_1, \cdots, y_n)$, where $y_i \in Y$, $i = 1, \cdots, n$. Consider the functional defined on the product $W \times Y^n$,

$$e(x, y) = \inf_{u \in l_y} \| x - u \|_X,$$

i. e. the error in the best approximation of the element x from the linear space $l_y = \mathrm{span}\{y_1, \cdots, y_n\}$ of the elements $y_1, \cdots, y_n$. Consider the problem of approximation of the set W by a subspace of type l_y, $y \in Y^n$. We call

$$d(W, Y^n, X) = \inf_{y \in Y^n} \sup_{x \in W} e(x, y) \tag{1}$$

the Kolmogorov n-width. It follows Traub et al.[2] that the problem of computing n-widths is closely related to the information-based complexity of approximation. We assume that the set W is equipped with a Borel field of subsets $B(W)$, and let μ be a probability measure defined on $B(W)$. We will also consider in X the manifold of all n-dimensional subspaces to which a measure was introduced in the following way. Let Y be some subspace of X equipped with the Borel field $B(Y)$ and let ν' be a probability measure on $B(Y^n)$.

Set $Y^n = Y \times \cdots \times Y$. Let $B(Y^n)$ be the minimal σ-field containing $B(Y) \times \cdots \times B(Y)$. Then there exists a unique probability measure ν on $B(Y^n)$, such that

$$\nu(A_1 \times \cdots \times A_n) = \nu(A_1) \cdots \nu(A_n) \tag{2}$$

for all $A_1, \cdots, A_n \in B(Y)$. For a functional $g(y)$, defined on Y^n, one may introduce the function of distribution as follows:

$$\inf_{y \in Y_\varepsilon^n} g(y) \equiv \sup_{\nu(G) \leqslant \varepsilon} \inf_{x \in Y^n \setminus G} g(y), \quad \varepsilon \in [0, 1]. \tag{3}$$

Let $y = (y_1, \cdots, y_n) \in Y^n$ be a fixed element. Define the δ-distance of the space W from the subspace l_y by

$$e_\delta(W, y) \equiv \inf_{\mu(Q) \leqslant \delta} \sup_{x \in W \setminus Q} e(x, y) \equiv \sup_{x \in W_\delta} e(x, y). \tag{4}$$

We call

$$d_{\varepsilon, \delta}[(W, \mu), (Y^n, \nu), X] = \inf_{y \in Y_\varepsilon^n} e_\delta(W, y)$$
$$= \inf_{y \in Y_\varepsilon^n} \sup_{x \in W_\delta} e(x, y) \tag{5}$$

the Kolmogorov (n, ε, δ)-width of the space (W, μ) relative to the manifold (Y^n, ν) in the X-norm. As the classical n-width, probabilistic width quantifies the error of the best approximation. However, in the classical case, the error is defined by their worst case with respect to a given class. So it does not reflect the

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behavior of the error functional for the best approximation in the whole space. In other words, it does not give information about the measure of the elements in the class that can be approximated to some degree, in particular, the measure of the elements on which the supremum is attained with respect to the order. In the probabilistic approach, the error is defined by the worst case performance on a subset of measure at least $1-\delta$, so $d_{n,\delta}$ can be understood as the μ-distribution of the best approximation on all subsets of W which reflects the intrinsic structure of the class. Therefore, a probabilistic case setting, as compared with the worst case setting, allows one to give deeper analysis of the smoothness and approximation for the function class.

The results concerning the calculation of n-widths of the smooth function classes equipped with some given measure are contained in Refs. [1~6].

Based on the Kolmogorov (n, ε, δ)-widths one can construct the function of distribution of values of the functional $e(x, y)$ in investigating the approximation of the whole set W. Such an approximation method is called nonadaptive. While in application it is necessary to solve the problem of approximation of any individual element by choosing the best subspace from some set of subspaces. Such methods will be called adaptive. We call

$$d_{\varepsilon,\delta}^{ad}[(W,\mu),(Y^n,\nu),X] = \sup_{x\in W_\delta}\inf_{y\in Y_\varepsilon^n} e(x,y), \tag{6}$$

the adaptive (n, ε, δ)-width of the space (W, μ), which relates to the manifold (Y^n, ν) in the X-norm. From the inequality

$$\sup_{x\in W\setminus Q}\inf_{y\in Y_\varepsilon^n} e(x,y) \leqslant \inf_{y\in Y_\varepsilon^n}\sup_{x\in W\setminus Q} e(x,y),$$

where Q is any subset of W with measure $\mu(Q)\leqslant\delta$, one can derive that the following relation

$$d_{\varepsilon,\delta}^{ad}[(W,\mu),(Y^n,\nu),X] \leqslant d_{\varepsilon,\delta}[(W,\mu),(Y^n,\nu),X] \tag{7}$$

between the Kolmogorov and adaptive (n, ε, δ)-widths holds.

For $1\leqslant p\leqslant\infty$, denote by L_p the Banach space consisting of all 2π-periodic measurable functions with the finite norm

$$\|x\|_p = \left(\int_0^{2\pi}|x(t)|^p dt\right)^{1/p}, \quad 1\leqslant p<\infty,$$

and with the usual change to sup when $p=\infty$.

Considering the Hilbert space L_2 with inner product

$$\langle x, y\rangle = \frac{1}{2\pi}\int_0^{2\pi} x(t)\bar{y}(t)dt, \quad x, y\in L_2,$$

and supposing x has the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e_k(t), \quad e_k(t) := \exp(ikt),$$

we define the Weil r-fractional derivative $(r\in\mathbb{R})$ as

$$x^{(r)}(t) = \sum_{k\in\mathbb{Z}\setminus\{0\}} (ik)^r c_k \exp(ikt),$$

$$(ik)^r = |k|^r \exp\left(\frac{\pi i}{2}\operatorname{sign} r\right).$$

1 Main results

Denote by W_2^r, $r>0$, the well-known Sobolev space consisting of all functions $x\in L_2$ with mean value $c_0=0$ and semi-norm $\|x\|_{W_2^r}^2 = \langle x^{(r)}, x^{(r)}\rangle$, which is a Hilbert space with the inner product defined by

$$\langle x, y\rangle_1 = \langle x^{(r)}, y^{(r)}\rangle.$$

Equip W_2^r with a zero mean Gaussian measure μ whose correlation operator C_μ has eigenfunctions $e_k=\exp(ik(\cdot))$ and eigenvalues $\lambda_k = a|k|^{-s}$, $a>0$, $s>1$, i.e.

$$C_\mu e_k = \lambda_k e_k, \quad k\in\mathbb{Z}\setminus\{0\}.$$

Let y_k, $k=\pm1, \pm2, \cdots$, be an orthonormal system in the space W_2^r, $\sigma_k = \langle C_\mu y_k, y_k\rangle$, and $\mathcal{D}$ be any Borel subset in $\mathbb{R}^{m-n+1}(m>n)$, then the measure of the cylindrical subsets G in the space W_2^r given by

$$G = \{x\in W_2^r: \langle x, y_n^{(-r)}\rangle_1, \cdots, \langle x, y_m^{(-r)}\rangle_1 \in \mathcal{D}\}$$

is equal to

$$\mu(G) = \prod_{k=n}^{m} 2(\pi\sigma_k)^{-1/2} \cdot \int_{\mathcal{D}} \exp\left(-\frac{1}{2}\sum_n^m \sigma_k^{-1}u_k^2\right) du_n\cdots du_m. \tag{8}$$

Let $Y^n = L_2\times\cdots\times L_2$ be the product of n copies of the space $L_2 = W^0$ and let $\nu = \mu_0\times\cdots\times\mu_0$ be the Gaussian measure on Y^n. Considering the approximation functional on $W_2^r\times Y^n$

$$e(x, y) = \inf_{u\in l_y}\|x-u\|_{L_q}, \tag{9}$$

the Kolmogorov and adaptive (n, ε, δ)-widths are defined on the basis of this functional.

It is known that if $r>\max\{0, 1/2-1/q\}$, then the space W_2^r can be imbedded into the space L_q, $1\leqslant q\leqslant\infty$ (see Ref. [7]). Let c, c_i, $i=0,1,\cdots$, be positive constants depending only on the parameters r, q, a and s. For two positive functions $a(y)$ and $b(y)$, $y\in\mathscr{D}$, the notation $a(y)\asymp b(y)$ or $a(y)\ll b(y)$ means that there exist constants c_1, c_2, or c such that $c_1\leqslant a(y)/b(y)\leqslant c_2$ or $a(y)\leqslant cb(y)$ for all $y\in\mathscr{D}$.

The main results of calculating the Kolmogorov and adaptive (n,ε,δ)-widths of the Sobolev space W_2^r equipped with a Gaussian measure in the L_1-norm and L_∞-norm are as follows.

Theorem 1. Let $r>1/2$, $q=1$ or ∞, $s>1$, $a>0$. Then for all $n=0,1,\cdots$, $\delta\in(0,1/2]$, $\varepsilon\in[0,1-2^{-n}]$, the Kolmogorov (n,ε,δ)-width of the space (W_r^2,μ) related to the manifold (Y^n,ν) in the L_q norm has the asymptotic value

$$d_{\varepsilon,\delta}[(W_2^r,\mu),(Y^n,\nu),L_q]\asymp\frac{1+\sqrt{(1/n)\ln(1/\delta)}}{n^{r+(s-1)/2}}.$$

Theorem 2. Under the conditions of Theorem 1 for all $\delta\in(0,1/2]$, $\varepsilon\in[\delta',1-2^{-n}]$, where $\delta'=\min\{\delta,2^{-n}\}$, the adaptive (n,ε,δ)-width has the asymptotic value

$$d_{\varepsilon,\delta}^{ad}[(W_2^r,\mu),(Y^n,\nu),L_q]\asymp\frac{1+\sqrt{(1/n)\ln(1/\delta)}}{n^{r+(s-1)/2}}.$$

Remark 1. In the case $1<q<\infty$, Theorems 1 and 2 have been investigated by Maiorov[1], and he has conjectured that it is possible to generalize his result to the case $q=1$, and $q=\infty$. In this paper, we will prove the conjecture, and determine the tight order of the Kolmogorov (n,ε,δ)-widths $d_{\varepsilon,\delta}[(W_2^r,\mu),(Y^n,\nu),L_q]$ and adaptive (n,ε,δ)-widths $d_{\varepsilon,\delta}^{ad}[(W_2^r,\mu),(Y^n,\nu),L_q]$ in the case $q=1$ and $q=\infty$.

2 Proofs of the main results

Let l_p^m be the m-dimensional space of vectors $\boldsymbol{x}=(x_1,\cdots,x_m)\in\mathbb{R}^m$ with the usual norm. Denote by $B_p^m(\rho)=\{\boldsymbol{x}\in l_p^m:\ \|x\|_p\leqslant\rho\}$ the ball with radius ρ in the space l_p^m. If $\rho=1$, then set $B_p^m=B_p^m(1)$. Consider the standard Gaussian measure γ_m on $\mathbb{R}^m$, which is defined by

$$\gamma_m(G)=(2\pi)^{-m/2}\int_G\exp\left(-\frac{1}{2}\sum_{i=1}^m x_i^2\right)\mathrm{d}x_1\cdots\mathrm{d}x_m,$$

where G is any Borel set in $\mathbb{R}^m$. Obviously $\gamma_m(\mathbb{R}^m)=1$. Denote by Y^n the space of all possible collections that consist of n vectors $\boldsymbol{y}=(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_n)$, $\boldsymbol{y}_i\in\mathbb{R}^m$, $i=1,\cdots,n$, and equip Y^n with measure $\gamma_{mn}=\gamma_m\times\cdots\times\gamma_m$.

Following Refs. [4] and [9], consider the two-sided sequence $\{\lambda_k(\cdot)\}_{k=-\infty}^{\infty}$ of continuous functions on the line $\mathbb{R}$. If $k\geqslant1$, let $\lambda_k(2^k)=1$ and $\lambda_k(u)=0$ for $u\notin[2^{k-1},2^{k+1}]$, and extend $\lambda_k(\cdot)$ linearly to the whole of $\mathbb{R}$; if $k\leqslant1$ let $\lambda_k(\cdot)=\lambda_{-k}(\cdot)$, choose the function $\lambda_0(\cdot)$ so that for all $u\in\mathbb{R}$

$$\sum_{k\in\mathbb{Z}}\lambda_k(\cdot)=1.$$

Construct a sequence of multiplicative operators which acts on the space L_∞. If $x(t)=\sum_{n\in\mathbb{Z}}c_ne_n(t)$, then set

$$(\Lambda_kx)(t)=\sum_{n\in Z}\lambda_k(n)c_ne_n(t),\quad e_n(t):=e^{int},\quad k\in\mathbb{Z}.$$

For $k>1$, denote by Δ_k the set of integers $\{2^{k-1},\cdots,2^{k+1}\}$, and for $k\leqslant-1$, let $\Delta_k=-\Delta_{-k}$, and $\Delta_0=\{-1,0,1\}$. We consider the following sequence of projection operators acting on L_∞:

$$(P_kx)(t)=\sum_{n\in\Delta_k}c_ne_n(t),\quad k\in\mathbb{Z}.$$

Let $m_k=\operatorname{card}\Delta_k$, i.e. $m_k=3\cdot2^{|k|-1}+1$ for $k\neq0$ and $m_0=3$. Denote by $\mathscr{T}^{m_k}(\Delta_k)$ the m_k-dimensional space of all trigonometric polynomials of the form

$$y(t)=\sum_{n\in\Delta_k}c_ne_n(t).$$

It forms a Banach space $\mathscr{T}_\infty^{m_k}(\Delta_k)$ consisting of the elements of $\mathscr{T}^{m_k}(\Delta_k)$ with the norm

$$\|y\|_{\mathscr{T}_\infty^{m_k}(\Delta_k)}=\max_{l=1,\cdots,m_k}|y(t_l)|,\quad t_l=2\pi l/m_k.$$

Let l_∞^m denote the m-dimensional normed space consisting of vectors $\boldsymbol{x}=(x_1,\cdots,x_m)\in\mathbb{R}^m$ with norm

$$\|x\|_\infty=\max_{1\leqslant i\leqslant m}|x_i|.$$

Now we establish an auxiliary discretization theorem that reduces the upper estimate of the (n,ε,δ)-width $d_{\varepsilon,\delta}[(W_2^r,\mu),(Y^n,\nu),L_\infty]$ to the corresponding finite-dimensional problem for the (n,ε,δ)-width $d_{\varepsilon_k,\delta_k}[(\mathbb{R}^{m_k},\nu_{m_k}),(\mathbb{R}^{m_kn_k},\gamma_{m_kn_k},l_\infty^{m_k})]$.

Theorem 3. Let $\rho=r+(s-1)/2$, ε, $\delta\in$

$[0,1]$, $k\in\mathbb{Z}$ and let the four sequences of numbers $\{m_k=3\cdot 2^{|k|-1}, m_0=3\}$, $\{n_k\}$, $\{\varepsilon_k\}$, and $\{\delta_k\}$ satisfy $0\leqslant n_k\leqslant m_k$, $\sum_{k\in\mathbb{Z}} n_k\leqslant n$, $\sum_{k\in\mathbb{Z}}\varepsilon_k\leqslant\varepsilon$, $\sum_{k\in\mathbb{Z}}\delta_k\leqslant\delta$ respectively. Then for the Kolmogorov (n,ε,δ)-widths of the space (W_2^r,μ) relative to manifold (Y^n,ν) in the L_∞-norm, we have

$$d_{\varepsilon,\delta}[(W_2^r,\mu),(Y^n,\nu),L_\infty]$$
$$\leqslant c_0\sum_{k\in\mathbb{Z}} m_k^{-\rho} d_{\varepsilon_k,\delta_k}[(\mathbb{R}^{m_k},\gamma_{m_k}),(\mathbb{R}^{m_k n_k},\gamma_{m_k n_k}, l_\infty^{m_k})].$$

Proof. Introduce the interpolation polynomial operator $y(t)$ assigned to a vector $(y_1,\cdots,y_{m_k})$ in $\mathcal{T}^{m_k}$ such that $y(t_l)=y_l$, $l=1,\cdots,m_k$, and consider the polynomials in $\mathcal{T}_\infty^{m_k}(\Delta_k)$

$$\varphi_l(t)=\sum_{n\in\Delta_k}e_n(t-t_l),\quad l=1,\cdots,m_k.$$

Obviously these polynomials are orthogonal in L_2. For any function $x\in W_2^r$, we have

$$(P_kD^rx)(t_l)=\langle D^rx,\varphi_l\rangle,\quad l=1,\cdots,m_k.$$

Let $\sigma_{kj}=\langle C_\mu\varphi_{kj},\varphi_{kj}\rangle$, $j=1,\cdots,m_k$, where C_μ is the correlation operator of measure μ. Then

$$\sigma_{k,1}=\cdots=\sigma_{k,m_k}\equiv\sigma_k,\quad \sigma_k\asymp m_k^{1-s}.$$

From the definition of operator C_μ, it follows that

$$\sigma_k=\langle C_\mu\varphi_{kl},\varphi_{kl}\rangle=a\sum_{l=1}^{\infty}l^{-s}|\langle\varphi_{kl},e_l\rangle|$$
$$=a\sum_{l\in\Delta_k}l^{-s}\asymp m_k^{1-s}.$$

For any $k\in\mathbb{Z}$, let $G_k'\subset\mathbb{R}^{m_k n_k}$ and $Q_k'\subset\mathbb{R}^{m_k}$ be sets such that

$$\gamma_{m_k n_k}(G_k')\geqslant 1-\varepsilon_k,\quad \gamma_{m_k}(Q_k')\leqslant\delta_k,$$

and for $(n_k,\varepsilon_k,\delta_k)$-width of the space $\mathbb{R}^{m_k}$

$$d_{\varepsilon_k,\delta_k}[(\mathbb{R}^{m_k},\gamma_{m_k}),(\mathbb{R}^{m_k n_k},\gamma_{m_k n_k}),l_\infty^{m_k}]$$
$$=\inf_{y\in G_k'}\sup_{x\in\mathbb{R}_{m_k}\setminus Q_k'}e(x,y,l_\infty^{m_k}).\qquad(10)$$

For any k and $i=1,\cdots,m_k$, define a functional on L_∞

$$f_{k_i}(x)=\langle\sigma_k^{-1/2}D^rx,\varphi_{ki}\rangle.$$

Obviously, we have

$$f_{k_i}(x)=\sigma_k^{-1/2}(D^kP_kx)(t_i).$$

Construct the operators from the space L_∞ to $\mathbb{R}^{m_k}$

$$F_k(x)=(f_{k1}(x),\cdots,f_{km_k}(x))$$

and construct the subset of W^r

$$Q_k=\{x\in W^r: F_kx\in Q_k'\}.$$

Analogously, construct an operator from Y^{n_k} to $\mathbb{R}^{m_k n_k}$

$$y=(y_1,\cdots,y_{n_k})\to\Phi_ky$$
$$=(f_{k_i}(y_j))_{i=1,\cdots,m_k;j=1,\cdots,n_k},$$

and consider the subset in Y^{n_k} given by

$$G_k=\{y\in Y^{n_k}:\Phi_ky\in G_k'\},$$

(for $k=0$, set $Q_0=\{0\}$ and $G_0=\varnothing$.) Let

$$G=\bigcap_{k\in\mathbb{Z}}G_k,\quad Q=\bigcup_{k\in\mathbb{Z}}Q_k.$$

Then $\nu(G)\geqslant 1-\varepsilon$, and $\mu(Q)\leqslant\delta$.

For any element $z\in L_\infty$, by Lemma 6 in Ref. [4] (see also Ref. [9]), we have $z=\sum_{k\in\mathbb{Z}}\Lambda_kz$ in the case $z\in L_\infty$ and

$$e(\Lambda_kx,\Lambda_ky,L_\infty)$$
$$\leqslant c_0m_k^{-r}e(D^rP_kx,D^rP_ky,T_\infty^{m_k}(\Delta_k)).\qquad(11)$$

By virtue of the relation

$$e(D^rP_kx,D^rP_ky,T_\infty^{m_k}(\Delta_k))=\sqrt{\sigma_k}e(F_kx,\Phi_ky,l_\infty^{m_k}),\qquad(12)$$

with $\sigma_l=\langle C_\mu\varphi_l,\varphi_l\rangle$, $l=1,\cdots,m_k$, it is clear that $\sigma_1=\cdots=\sigma_{m_k}$, and

$$\sigma_1=a\sum_{n\in\mathbb{Z}\setminus\{0\}}|n|^{-s}|\langle\varphi_1,e_n\rangle|$$
$$=a\sum_{n\in\Delta_k}|n|^{-s}\asymp am_k^{1-s}.$$

Hence we get $m_k^{-r}\sqrt{\sigma_k}\asymp m_k^{-\rho}$. Using the definitions of the sets G and Q, we obtain from Lemma 6 in Refs. [4] and (10)~(12)

$$\inf_{y\in G}\sup_{x\in W_2^r\setminus Q}e(x,y,L_\infty)$$
$$\leqslant c_0\sum_{k\in\mathbb{Z}}m_k^{-\rho}\inf_{y\in G}\sup_{x\in W_2^r\setminus Q}e(F_kx,\Phi_ky,l_\infty^{m_k})$$
$$\leqslant c_0\sum_{k\in\mathbb{Z}}m_k^{-\rho}\inf_{y'\in G_k'}\sup_{x'\in W^r\setminus Q_k'}e(x',y',l_\infty^{m_k})$$
$$\leqslant c_0\sum_{k\in\mathbb{Z}}m_k^{-\rho}d_{\varepsilon_k,\delta_k}[(\mathbb{R}^{m_k},\gamma_{m_k}),(\mathbb{R}^{m_k n_k},\gamma_{m_k n_k}),l_\infty^{m_k}],\qquad(13)$$

from which and the definitions of G, T, $\{G_k\}$, $\{T_k\}$, we estimate continuously, and obtain

$$d_{\varepsilon,\delta}[(W_2^r,\mu),(Y^n,\nu),L_\infty]$$
$$\leqslant c_0\sum_k m_k^{-\rho}d_{\varepsilon_k,\delta_k}[(\mathbb{R}^{m_k},\nu_{m_k}),(\mathbb{R}^{m_k n_k},\gamma_{m_k n_k}),l_\infty^{m_k}].$$

The proof of Theorem 3 is complete.

Theorem 4. If $m>n$, 0, ε, $\delta\in[0,1]$, $\omega:=r+(s+1)/2$, then

$$d_{\varepsilon,\delta}^{ad}[(W_2^r,\mu),(Y^n,\nu),L_1]$$

$$\geqslant cm^{-\omega}d_{\varepsilon,\delta}^{ad}[(\mathbb{R}^m,\gamma_m),(\mathbb{R}^{mn},\gamma_{mn}),l_1^m].$$

Proof. Without loss of generality, we prove only for the case $\varepsilon=0$. Let $m>0$ be any integer, and let

$$s_l(t)=s_1(t-2\pi(l-1)/m),\quad l=1,\cdots,m,$$

where

$$s_1(t)=\begin{cases}(m/\pi)^{2r}t^r(2\pi/m-t)^r, & 0\leqslant t\leqslant 2\pi/m,\\ 0, & 2\pi/m<t<2\pi.\end{cases}$$

Then for all l, the quantities $\sigma_l=\langle C_\mu D^{-2r}s_l,s_l\rangle=\langle C_\mu D^{-r}s_l,D^{-r}s_l\rangle$ have the estimate

$$\sigma_l\asymp m^{-(2r+s+1)},$$

for $l=1,\cdots,m$. We define a functional on L_1

$$f_l(x)=\langle\sigma_l^{-1/2}x,s_l\rangle,\tag{14}$$

and let

$$(l_y)^{\perp}=\{z\in L_\infty:\langle z,u\rangle=0,\quad\forall u\in l_y\}.$$

By the Nikolskii's Duality Theorem (Pages 13～15 in Ref. [10]),

$$e(x,y,L_1)=\sup\{|\langle x,z\rangle|:z\in(l_y)^{\perp},\|z\|_{L_\infty}\leqslant 1\},\quad\forall y\in Y^n.$$

Consider in $\mathbb{R}^m$ the subspace

$$(\bar{l}_y)=\left\{u=(u_1,\cdots,u_m)\in\mathbb{R}^m:\sum_{l=1}^m u_ls_l(\cdot)\in(l_y)^{\perp}\right\}.\tag{15}$$

Since $\|s_l\|_\infty=1$, $l=1,\cdots,m$, and the supports of the s_l are disjoint, it follows from (15) that

$$\begin{aligned}e(x,y,L_1)&\geqslant\sup\left\{\left|\sum_{l=1}^m u_l\langle x,s_l\rangle\right|:u\in(\bar{l}_y)^{\perp},\|u\|_\infty\leqslant 1\right\}\\&=\sigma^{1/2}\sup\left\{\left|\sum_{l=1}^m u_lx_l\right|:u\in(\bar{l}_y)^{\perp},\|u\|_\infty\leqslant 1\right\}\\&=\sigma^{1/2}e(x',y',l_1^m),\end{aligned}$$

where $x'=(x_1,\cdots,x_m)$ and $x_l=f_l(x)$, $l=1,\cdots,m$ defined by (14) and l_y' is the orthogonal complement of $(\bar{l}_y)^{\perp}$. Let the set $Q\subset W_2^r$ satisfy $\mu(Q)\leqslant\delta$, and

$$d_{0,\delta}^{ad}[(W_2^r,\mu),(Y^n,\nu),L_1]=\sup_{x\in W_2^r\setminus Q}\inf_{y\in Y^n}e(x,y,L_1).\tag{16}$$

We construct the operator from L_1 to $\mathbb{R}^m$

$$F(x)=(f_1(x),\cdots,f_m(x))$$

and let $Q'=FQ$. So from the definition of the measure μ, $\gamma_m(Q')=\mu(Q)\leqslant\delta$, and by the definitions of the operators F, it follows that $F(W_2^r\setminus Q)=\mathbb{R}^m\setminus Q'$. Therefore, we have

$$\begin{aligned}&\sup_{x\in W_2^r\setminus Q}\inf_{y\in Y^n}e(x,y,L_1)\\&\geqslant m^{-\omega}\sup_{x'\in\mathbb{R}^m\setminus Q'}\inf_{y'\in\mathbb{R}^{mn}}e(x',y',l_1^m)\\&\geqslant m^{-\omega}d_{0,\delta}^{ad}[(\mathbb{R}^m,\gamma_m),(\mathbb{R}^{mn},\gamma_{mn}),l_1^m],\end{aligned}$$

Theorem 4 is proved.

Now we prove our main results.

Proof. (for Theorems 1 and 2). By the inequality (7), it is sufficient to prove the upper estimate for Theorem 1 and the lower estimate for Theorem 2. Assume that $n=2^{k'}$, $m_k=3\cdot 2^{|k|-1}+1$, and

$$\varepsilon_k=\begin{cases}0, & |k|\neq k',\\ \varepsilon/2, & \text{otherwise},\end{cases}$$

$$n_k=\begin{cases}m_k, & |k|<k'-1,\\ n2^{k'-k}, & |k|\geqslant k'-1,\end{cases}$$

$$\delta_k=\begin{cases}0, & |k|<k'-1,\\ \delta 2^{k'-k}, & |k|\geqslant k'-1.\end{cases}$$

Obviously, ε_k, n_k, and δ_k satisfy

$$\sum_{k\in Z}\varepsilon_k=\varepsilon,\quad\sum_{k=-\infty}^{\infty}n_k\ll n\ \text{and}\ \sum_{k=-\infty}^{\infty}\delta_k\ll\delta.$$

Therefore by Theorem 3 in this paper and Theorem 5 in Ref. [1], we can obtain

$$\begin{aligned}&d_{\varepsilon,\delta}[(W_2^r,\mu),(Y^n,\nu),L_\infty]\\&\leqslant c_0\sum_{k=-\infty}^{\infty}m_k^{-\rho}d_{\varepsilon_k,\delta_k}[(\mathbb{R}^{m_k},\gamma_{m_k}),(\mathbb{R}^{m_kn_k},\gamma_{m_kn_k}),l_\infty^{m_k}]\\&\ll\sum_{k>k'}2^{-\rho k}\sqrt{(1+1/n_k\ln(1/\delta_k)\ln(m_k-n_k))}.\end{aligned}$$

Substituting the values of m_k, n_k, and δ_k into the above formula, we estimate countinuosly,

$$\begin{aligned}&\sum_{k>k'}m_k^{-\rho}\sqrt{(1+1/n_k\ln(1/\delta_k))\ln(m_k-n_k)}\\&\ll\sum_{k>k'}2^{-\rho k}(\ln(m_k-n_k))^{1/2}\\&\quad+\sum_{k>k'}(1/n_k\ln(1/\delta_k)\ln(m_k-n_k))^{1/2}\\&\ll\sum_k 2^{-\rho k}(k-k')^{1/2}+n^{-1}\sum_k 2^{(-\rho+1/2)k}\\&\quad\cdot[(k-k'+\ln(1/\delta))(k-k')]^{1/2}.\end{aligned}$$

Substituting $k-k'$ by k in the sums, we finally obtain

$$\begin{aligned}&d_{\varepsilon,\delta}[(W_2^r,\mu),(Y^n,\nu),L_\infty]\\&\ll 2^{-\rho k'}+n^{-1}2^{(-\rho+1/2)k'}\sqrt{\ln(1/\delta)}\\&\ll n^{-\rho}(1+\sqrt{1/n\ln(1/\delta)}).\end{aligned}$$

The upper estimate of Theorem 1 is completed. Now we proceed the lower estimate of Theorem 2. By virtue of Theorem 4 in this paper and Theorem 6 in Ref. [1], we have

$$\begin{aligned} & d^{ad}_{\epsilon,\delta}[(W^r_2,\mu),(Y^n,\nu),L_1] \\ & \gg m^{-(r+(s+1)/2)} d^{ad}_{\epsilon,\delta}[(\mathbb{R}^m,\gamma_m),(\mathbb{R}^{mn},\gamma_{mn}),l^m_1] \\ & \gg n^{-(r+(s+1)/2)} \sqrt{(2n)^2+2n\ln(1/\delta)} \\ & \gg n^{-(r+(s-1)/2)}(1+\sqrt{1/n\ln(1/\delta)}), \end{aligned}$$

for $m=2n$. Now we obtain the lower bound of Theorem 2.

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